

IMPROVED SOLUTION OF K-V ENVELOPE EQUATIONS FOR TRANSPORT IN QUADRUPOLE CHANNELS*

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The envelope equations for a K-V beam with space charge are analyzed systematically by an ϵ expansion followed by integrations. The focusing field profile as a function of axial length is assumed to be symmetric but otherwise arbitrary. Given the beam current, emittance, and peak focusing field, we find the envelopes $a(s)$ and $b(s)$ and obtain $\langle a \rangle$, a_{\max} , and the phase advances σ and σ_0 . Explicit results are presented for various truncations of the expansion. The zeroth-order results correspond to those from the well-known smooth approximation; the same convenient format is retained for the higher-order cases. The first-order results, involving single correction terms, give 3 to 10 times improvement in accuracy and are good to $\sim 1\%$ at $\sigma_0 = 70^\circ$. Third order gives a factor of 10–30 improvement over the smooth approximation and derived quantities are accurate to $\sim 1\%$ at $\sigma_0 = 112^\circ$. The first-order expressions are convenient design tools. They lend themselves to variable energy problems and have been applied to the design, construction, and testing of ESQ accelerators at LBL.

Keywords: Beam transport, collective effects, particle dynamics

1 K-V ENVELOPE EQUATIONS

A non-relativistic beam with a uniform density (K-V distribution) transported by a series of linear symmetric quadrupoles is described by the paraxial equations for the envelopes a and b :

$$a'' = -K(z)a + \frac{\epsilon^2}{a^3} + \frac{2Q}{a+b} \quad (1)$$

$$b'' = +K(z)b + \frac{\epsilon^2}{b^3} + \frac{2Q}{a+b}, \quad (2)$$

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where $K(z)$ represents the alternating quadrupole gradient and ϵ is the emittance (we assume $\epsilon_x = \epsilon_y$). Q is the normalized perveance, defined nonrelativistically by $Q = (4\pi\epsilon_0)^{-1}(m/2q)^{1/2}IV^{-3/2}$, with m the mass, I the beam current, and qV the beam energy.

1.1 Review of Smooth Approximation Formulas

Before presenting our own improved results, we recollect the familiar smooth approximation formulas¹⁻³ and introduce more of our notation. The standard formula for phase advance σ_0 in the smooth approximation is¹

$$\frac{\sigma_0^2}{(2L)^2} = \frac{1}{2L} \int_0^{2L} K(z) \rho(z) dz \quad (3)$$

where Equation (31) in Ref. 1 has been converted to our notation, with cell length $S \rightarrow 2L$, ripple $\delta_x(s) \rightarrow \rho(z)$, $\mu_0 \rightarrow \sigma_0$, and $\kappa_x(s) \rightarrow K(z)$. The standard matching equation in the smooth approximation is

$$\frac{\sigma_0^2}{(2L)^2} = \frac{\epsilon^2}{A^4} + \frac{Q}{A^2},$$

which is Equation (37) in Ref. 1 with mean radius $\bar{R} \rightarrow A$ and $K \rightarrow Q$. For Equation (3), Ref. 1 does not explain how $\rho(z)$ is to be calculated, but as shown below [§2.2], $\rho(z)$ in lowest order is obtained by integrating $K(z)$ twice. Then, after an exchange of integrations, the right side of Equation (3) is $\langle [\int^z dz' K(z')]^2 \rangle$. We use this quantity often and name it K_{eff} , the effective focusing force:

$$\left\langle \left[\int^z dz' K(z') \right]^2 \right\rangle \equiv K_{\text{eff}} \quad (4)$$

where $\langle \dots \rangle$ means averaging over a cell of length $2L$. The zeroth-order matching equation is then

$$K^{\text{eff}} = \frac{\epsilon^2}{A^4} + \frac{Q}{A^2}. \quad (5)$$

Equation (3) in the form of (4) is also given in Ref. 3, Equation (10.92), which explicitly *defines* to lowest order $\sigma_0^2/(2L)^2 = \langle [\int^z dz' K(z')]^2 \rangle$. Therefore, in this paper

$$\sigma_{0\text{smooth}} = 2L K_{\text{eff}}^{1/2} \quad (6)$$

is called the smooth approximation for σ_0 .

The depressed tune σ is (Ref. 2, Equation (6)),

$$\sigma_{\text{smooth}} = 2L \frac{\epsilon}{A^2}, \quad (7)$$

where A^{-2} is obtained from the zero-order equation (5).

The smooth-approximation formulas are popular design tools for AG systems because they are simple and explicit. However, they become seriously inaccurate for applications with large focusing fields and large phase advances. Our simple generalizations derived below improve the accuracy by factors of 3 to 10 or more (see table and figures, Section 3) while retaining the above simple explicit formats.

2 SYSTEMATIC SOLUTION

We allow the form of $K(z)$ to be arbitrary except for the assumption of symmetry. We write $K(z) = Kh(z)$ with $K \equiv K(0)$ and $h(0) = 1$ and define the cell length as $2L$. We assume the quadrupole form factor h is symmetric about $z = 0$, periodic over a cell length and antiperiodic over a half-cell length:

$$h(-z) = h(z), \quad h(z - 2L) = h(z), \quad h(z - L) = -h(z). \quad (8)$$

Therefore $h(L/2) = 0$, h is antisymmetric about $L/2$, and $\langle h(z) \rangle = 0$. In the following, we start all our integrations at $z = 0$, where $h = 1$.

2.1 Expansion about Mean Radius

We assume the beam is matched, so that $\langle a \rangle = \langle b \rangle \equiv A$ (the mean radius), and write $a(z) = A + \tilde{a}(z) = A(1 + \rho)$, where we define the ripple ratio

$$\rho(z) \equiv \frac{\tilde{a}(z)}{A}.$$

The assumption of quadrupole symmetry means that $\tilde{b}(z) \approx -\tilde{a}(z)$, so that $a + b \approx 2A$. Actually, there is a correction term, discussed below Equation (23), which we drop without affecting the results. With or without that term, the coupling between Equations (1) and (2) is eliminated:

$$a'' = -K(z)a + \frac{\epsilon^2}{a^3} + \frac{Q}{A}. \quad (9)$$

(After solving for a , the solution for b is obtained by changing the signs of terms containing odd powers of K .) Substituting $A(1 + \rho)$ for $a(z)$ in (9), expanding, and dividing by A , we have

$$\rho'' = -Kh(z) - Kh\rho + \frac{\epsilon^2}{A^4}(1 - 3\rho + 6\rho^2 \dots) + \frac{Q}{A^2}. \quad (10)$$

Averaging,

$$0 = -K\langle h\rho \rangle + \frac{\epsilon^2}{A^4}(1 + 6\langle \rho^2 \rangle \dots) + \frac{Q}{A^2}. \quad (11)$$

Subtracting,

$$\rho'' = -Kh(z) - K\{h\rho\} - \frac{3\epsilon^2}{A^4}(\rho - 2\{\rho^2\} \dots), \quad (12)$$

where the operator $\{\dots\}$ gives just the oscillatory part of a function:

$$\{\phi\} \equiv \phi - \langle \phi \rangle. \quad (13)$$

If we assume that a never vanishes, then $\rho < 1$ and the above Taylor expansion converges; (11) and (12) taken together have exactly the same content as the original equation, (9). Here, we keep only the terms shown, but Ref. 4 includes terms through ρ^6 .

2.2 Systematic Solution: Periodic Part

We follow Courant and Snyder in their treatment of the Hill equation.⁵ They regard the focusing coefficient $K(z)$ as “small in some sense,” put $K(s) = \frac{1}{2}\epsilon g(s)$, and expand their β function ($\sim a^2$) in powers of the “smallness parameter” ϵ . Our treatment differs in that we include space charge and must work with $a = A(1 + \rho)$ instead of their β function.⁶

$$a(z) = A(1 + \rho(z)) \equiv A(1 + \epsilon f_1(z) + \epsilon^2 f_2(z) + \epsilon^3 f_3(z) + \dots). \quad (14)$$

In Ref. 4 we show how terms through $\epsilon^7 f_7$ are feasibly included, but here we use only the three terms shown in (14).

As in Ref. 5, our basic small quantity is the focusing strength K , and we write

$$K \equiv \epsilon k. \quad (15)$$

From (5), we expect that $K^2 L^2 \cdot \text{Cnst} \approx \epsilon^2 / A^4 + Q / A^2$. The terms on the right can be no larger than $\epsilon^2 K^2 L^2$, so we give them ϵ^2 ordering and define α and q by:

$$\frac{3 \epsilon^2}{A^4} \equiv \epsilon^2 \alpha, \quad (16)$$

$$\frac{Q}{A^2} \equiv \epsilon^2 q. \quad (17)$$

(We assume that either of these terms could dominate, i.e., σ is in the range $0 \leq \sigma \leq \sigma_0$.) The factor A — from (14) — will be determined later.

We insert (14)–(17) into (12). Through ϵ^3 ,

$$\epsilon f_1'' + \epsilon^2 f_2'' + \epsilon^3 f_3'' = -\epsilon k h(z) - \epsilon^2 k \{h f_1\} - \epsilon^3 k \{h f_2\} - \epsilon^3 \alpha f_1.$$

Equating like powers of ϵ , $f_1'' = -k h(z)$, $f_2'' = -k \{h f_1\}$, and $f_3'' = -\alpha f_1 - k \{h f_2\}$. Integrating,

$$\begin{aligned} f_1 &= -k \iint h \\ f_2 &= k^2 \iint \{h g\}, \\ f_3 &= +\alpha k \iint g - k^3 \iint \{h \delta\}, \end{aligned}$$

where

$$g \equiv \iint h. \quad (18)$$

The small term

$$\delta(z) \equiv \iint \{h g\} \quad (19)$$

has fundamental frequency double that of the focusing lattice. Our operator \iint is defined to give just the oscillatory part of the repeated integral:

$$\iint \Psi \equiv \left\{ \int_0^z dz' \int_0^{z'} \Psi(z'') dz'' \right\}.$$

Note: For an operand Φ with the symmetries of Equation (8) — e.g., $g(z)$,

$$\iint \Phi = \int_{L/2}^z dz' \int_0^{z'} \Phi(z'') dz''.$$

For an operand such as $\{hg\}$ which lacks this symmetry, one could adjust the lower limit, but it is easier to subtract the average as in (13).

A feature of our ordering is that f_1, f_3 , etc., turn out to have only odd harmonics and odd powers of k , while f_2 , etc., have only the even cases. Thus

$$\rho = \frac{\tilde{a}(z)}{A} = -\varepsilon kg + \varepsilon^2 k^2 \delta + \varepsilon^3 \alpha k \iint g - \varepsilon^3 k^3 \iint \{h\delta\} + \dots, \quad (20)$$

$$\frac{\tilde{b}(z)}{A} = +\varepsilon kg + \varepsilon^2 k^2 \delta - \varepsilon^3 \alpha k \iint g + \varepsilon^3 k^3 \iint \{h\delta\} + \dots$$

Defining the leading-order ripple,

$$\rho_0(z) \equiv -\varepsilon kg = -Kg = -K \iint h, \quad (21)$$

and using (15) and (16), we have, finally,

$$\rho = \frac{\tilde{a}(z)}{A} = +\rho_0 + K^2 \delta(z) - \frac{3\varepsilon^2}{A^4} \iint \rho_0 - K^3 \iint \{h\delta\} + \dots, \quad (22)$$

$$\frac{\tilde{b}(z)}{A} = -\rho_0 + K^2 \delta(z) + \frac{3\varepsilon^2}{A^4} \iint \rho_0 + K^3 \iint \{h\delta\} + \dots \quad (23)$$

Note: For large focusing strengths, the double-frequency term $K^2 \delta$ becomes significant: e.g., $K^2 \delta \approx 0.025$ for $\sigma_0 = 120^\circ$. Noting that $a + b = 2A(1 + K^2 \delta)$, one might think it necessary to include the correction factor $(1 - K^2 \delta)$ on the Q term in Equation (9). In Ref. 4 this is done and shown to affect the results only in higher order: for σ_0 as large as 120° , the correction contributes at most 0.04% to the maximum radius and nothing at all to the matching equation.

The matching equation, derived below, gives the mean radius A needed for (22) and (23).

2.3 Systematic Solution: Average Part, Matching Equation

Using (15)–(17), Equation (11) becomes

$$\varepsilon k \langle h\rho \rangle = \varepsilon^2 \frac{\alpha}{3} + 2\varepsilon^2 \alpha \langle \rho^2 \rangle + \varepsilon^2 q; \quad (24)$$

$\rho(z)$ is given by (20) so that, to order ε^4 ,

$$2\varepsilon^2 \alpha \langle \rho^2 \rangle = 2\varepsilon^4 \alpha k^2 \langle g^2 \rangle. \quad (25)$$

By the above-mentioned k parity, $\langle hf_2 \rangle = 0$, so

$$\begin{aligned} \varepsilon k \langle h\rho \rangle &= -\varepsilon^2 k^2 \langle h \int f h \rangle + \varepsilon^4 \alpha k^2 \langle h \int f g \rangle - \varepsilon^4 k^4 \langle h \int f \{h\delta\} \rangle \\ &= +\varepsilon^2 k^2 \langle [\int h]^2 \rangle + \varepsilon^4 \alpha k^2 \langle g^2 \rangle + \varepsilon^4 k^4 \langle [\int \{hg\}]^2 \rangle. \end{aligned}$$

We rearranged integrations using the $h(z)$ symmetries, Equation (8). For example, $-\langle h \int f h \rangle = \langle [\int h]^2 \rangle$, with notation $\int \Psi \equiv \int_0^z \Psi(z') dz'$.

The $\langle g^2 \rangle$ term cancels half of (25), and the matching equation through ε^4 is

$$\varepsilon^2 k^2 \langle [\int h]^2 \rangle + \varepsilon^4 k^4 \langle [\int \{hg\}]^2 \rangle = \varepsilon^2 \frac{\alpha}{3} + \varepsilon^4 \alpha k^2 \langle g^2 \rangle + \varepsilon^2 q. \quad (26)$$

For $\langle [\int \{hg\}]^2 \rangle$ we expand: $h(z) = h_1 \left[\cos \frac{\pi z}{L} + \frac{c_3}{3} \cos \frac{3\pi z}{L} + \dots \right]$ and find

$$\langle [\int \{hg\}]^2 \rangle \approx \frac{1}{8} \langle g^2 \rangle \left(1 + \frac{20}{27} c_3 + \dots \right) \langle [\int h]^2 \rangle.$$

(For FODO with occupancy $\eta = 0.5$, $c_3 = 1$.) We define the LHS of (26) as

$$K_I^{\text{eff}} \equiv K^{\text{eff}} \left[1 + \frac{1}{8} (1 + 0.74 c_3) \langle \rho_0^2 \rangle \right], \quad (27)$$

where we recall $K^{\text{eff}} = K^2 \langle [\int h]^2 \rangle$, and where we use the abbreviation

$$\langle \rho_0^2 \rangle \equiv K^2 \langle g^2 \rangle = K^2 \langle [\int f h]^2 \rangle, \quad (28)$$

the mean-square ripple to leading order. For the right side of (26), we define

$$\varepsilon_I^2 \equiv \varepsilon^2 (1 + 3 \langle \rho_0^2 \rangle). \quad (29)$$

Using (15)–(17) in reverse to remove ε 's, our matching equation (26) becomes

$$K_I^{\text{eff}} = \frac{\varepsilon_I^2}{A^4} + \frac{Q}{A^2} \quad (30)$$

or

$$A^2 = \frac{1}{2K^{\text{eff}}} \left[Q + \left(Q^2 + 4 \varepsilon_I^2 K_I^{\text{eff}} \right)^{1/2} \right], \quad (31)$$

giving the coefficient for (14). Note that (31) is valid in the limiting cases $Q = 0$ or $\varepsilon = 0$. In Equation (30) we have retained the convenient form of the smooth

approximation (5) by introducing the one-term corrections denoted by subscripts I . (In Ref. 4 we go to higher order and include upto six correction terms.)

Envelopes: For a matched beam, $a(z) = A + \tilde{a}(z)$ and $b(z) = A + \tilde{b}(z)$, where one uses (31), (22) and (23).

2.4 Phase Advances

Depressed Tune: From the well-known phase-amplitude result,⁵ the phase advance per quadrupole cell of length $2L$ is $\sigma = \epsilon \int_0^{2L} a^{-2} dz$. Using the definition $a(z) \equiv A[1 + \rho(z)]$, expanding, and noting that the 2ρ term has zero average, one obtains $\sigma = 2L \epsilon A^{-2}(1 + 3\langle\rho^2\rangle \dots)$. To leading order, $\langle\rho^2\rangle \approx \langle\rho_0^2\rangle$, and we have

$$\sigma = 2L \frac{\epsilon}{A^2} (1 + 3\langle\rho_0^2\rangle). \quad (32)$$

This resembles the smooth approximation, Equation (7), but accuracy is improved here by the correction term and also by the more accurate calculation of A^{-2} .

Undepressed Tune: We set $Q = 0$ so that (30) becomes

$$K_I^{\text{eff}} = \frac{\epsilon_I^2}{A^4} = \frac{\epsilon^2}{A^4} (1 + 3\langle\rho_0^2\rangle)$$

with K_I^{eff} from (27). Combining this with (32) for $Q = 0$ ($\sigma \rightarrow \sigma_0$), we eliminate ϵA^{-2} to get

$$\sigma_0 = 2L (K_I^{\text{eff}})^{1/2} \left(1 + \frac{3}{2} \langle\rho_0^2\rangle \right). \quad (33)$$

[Cf. Equation (6).] We shall see below that the simple corrections in these formulas for σ and σ_0 provide 3 to 10 times greater accuracy than the smooth approximations.

We emphasize that all the above results apply to a general symmetric lattice⁴ with or without discontinuities in $K(z)$.

3 SPECIAL CASE: FODO LATTICE

3.1 Solution of Ripple Equation

For the FODO cell, it is not hard to write down the lowest-order ripple function $\rho_0(z)$ — see Ref. 4. Here, we just quote its maximum value. With η the occupancy

factor (i.e., quad length/ L),

$$\varepsilon f_1(0) = \rho_0^{\max} = K \int_0^{L/2} dz \int_0^z h(z') dz' = \frac{1}{8} \eta (2 - \eta) K L^2. \quad (34)$$

For $\varepsilon^2 f_2 = K^2 \int \{hg\}$, it is convenient to Fourier analyze — as under Equation (26) — because most harmonics make negligible contributions to the results. We find

$$\varepsilon^2 f_2 \approx \frac{1}{8} \rho_m^2 \left(1 + \frac{10}{27} c_3 \right) \cos 2 \frac{\pi z}{L}; \quad (35)$$

$\rho_m \equiv h_1 K L^2 / \pi^2$. Neglected terms in (35) would contribute less than 0.06% to the final result for a_{\max} .

For $\varepsilon^3 f_3$ we integrate the first term but use Fourier representation for the second. We quote just the maximum value:⁴

$$\varepsilon^3 f_3(0) = \frac{1}{16} \eta \left(1 - \frac{\eta^2}{2} + \frac{\eta^3}{8} \right) \frac{\varepsilon^2}{A^4} K L^4 + \frac{1}{16} \rho_m^3 \left(1 + \frac{10}{27} c_3 \right). \quad (36)$$

The maximum radius is found by adding up the above results:

$$a^{\max} = A[1 + \varepsilon f_1(0) + \varepsilon^2 f_2(0) + \varepsilon^3 f_3(0)]. \quad (37)$$

3.2 Matching Equation, Phase Advance, Transportable Current

For the FODO model, we obtain from the definition (§ 1.1)

$$K_{\text{eff}} = K^2 \left\langle [fh]^2 \right\rangle = \frac{1}{12} \eta^2 (3 - 2\eta) K^2 L^2, \quad (38)$$

which is used to calculate K_I^{eff} on the left side of the matching equation (30). On both sides of (30) there are correction terms involving $\langle \rho_0^2 \rangle$; from Equation (21) we find

$$\langle \rho_0^2 \rangle = \frac{1}{120} \eta^2 \left(\frac{5}{2} - \frac{5}{2} \eta^2 + \eta^3 \right) K^2 L^4 \quad (39)$$

for the hard-edge model. With these results, Equation (30) can be solved for the transportable current Q or the matched beam radius A . We also use (39) in (32) and (33) to get the phase advances σ_0 and σ shown in Table 1.

TABLE 1: Comparison of exact and analytic solutions of Equations (14)–(15) for FODO lattice with $L = 10$ cm, $\eta = 0.5$, quadrupole radius $a_Q = 1.75$ cm, $qV = 200$ kV, H^- ions.

Beam current (Amps)	ϵ_N (π rad -cm)	V_Q (kV)	Average radius (cm)				Maximum radius (cm)				Depressed tune σ (degrees)			
			Smooth		3rd order		0th order		1st order		Smooth		1st order	
			approx	Eq. (30)	Ref. 4	Exact	Eq. (33)	Eq. (36)	Ref. 4	Exact	approx	Eq. (31)	Ref. 4	Exact
0.8	0.001	20	1.143	1.138	1.137	1.137	1.493	1.506	1.506	1.507	0.04	0.05	0.05	0.05
0.7	0.15	20	1.074	1.070	1.069	1.069	1.403	1.416	1.416	1.417	7.21	8.24	8.37	8.37
0.6	0.50	20	1.049	1.051	1.052	1.052	1.370	1.396	1.398	1.400	25.22	28.46	28.98	28.97
0.4	1.00	20	1.060	1.076	1.083	1.084	1.384	1.442	1.455	1.459	49.40	54.28	55.46	55.47
0.1	1.20	20	0.979	1.004	1.019	1.023	1.278	1.359	1.392	1.398	69.56	74.86	76.47	76.50

Phase advance σ_0 (degrees)			
V_Q (kV)	Smooth	1st order	3rd order
	approx	Eq. (32)	Ref. 4
10	38.19	38.87	38.91
15	57.28	59.59	59.93
20	76.38	81.86	83.41
25	95.47	106.21	111.47

Exact	
sol'n	Ref. 4
38.91	38.91
59.90	59.90
83.37	83.37
112.24	112.24

3.3 Discussion of Table 1 and Figures 1 and 2

In Table 1, the lattice parameters, quadrupole voltage V_Q , beam current I , and normalized emittance ϵ_N are given quantities. Derived quantities are shown to various orders; n th-order means keeping terms in Equation (14) through ϵ^{2n+1} . First-order results for A , a^{\max} , σ and σ_0 are calculated from Equations (30), (37), (32) and (33), respectively, along with (38) and (39). The lattice parameters shown at the top of Table 1 are the same as for the MFE prototype ESQ accelerator,⁷ except that here the occupancy η is taken to be 0.5.

Table 1 compares the above analytic results with exact values obtained by numerical integration of Equations (1) and (2). For the A , a^{\max} and σ tables, the constant 20 kV focusing voltage V_Q produces a phase shift σ_0 of 83.4°; the beam parameters (I , ϵ) are adjusted to keep the beam radius roughly constant while σ varies widely. Table 1 also gives smooth-approximation results for A , σ , and σ_0 and the lowest-order result for a^{\max} .

Third-order results from Ref. 4 are also shown, with accuracy usually within a few parts per thousand.

The accuracy of the various orders of approximation in Table 1 can be seen at a glance in Figures 1 and 2.

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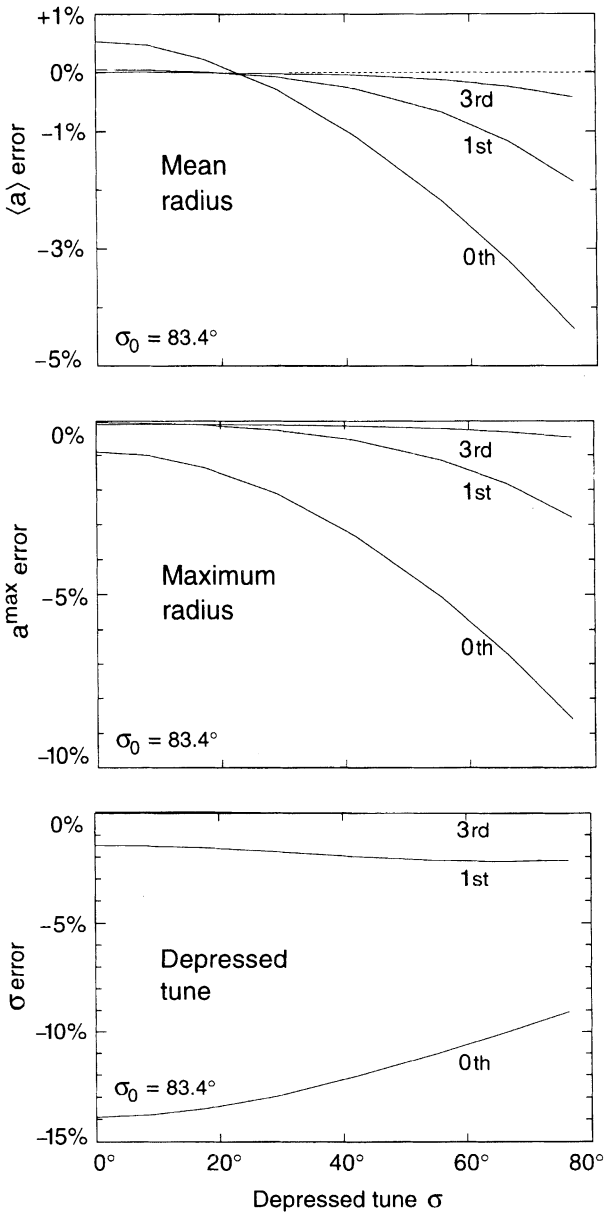


FIGURE 1: Accuracy of $\langle a \rangle$, a^{\max} , and σ to various orders (§3.3): 0th order from smooth approximation — Equations (5), (6), and (7); 1st order from Equations (30), (37), (32) and (33), etc.; 3rd order from Ref. 4. The quadrupole voltage V_Q , beam current I , and normalized emittance ϵ_N are input quantities. The lattice parameters (shown at the top of Table 1) are from the MFE ESQ accelerator⁷ with $\eta = 0.5$. V_Q is fixed at 20 kV, giving $\sigma_0 = 83.37^\circ$. As in Table 1, ϵ_N and I are varied so that σ (the abscissa) ranges between 0° and 83.32° .

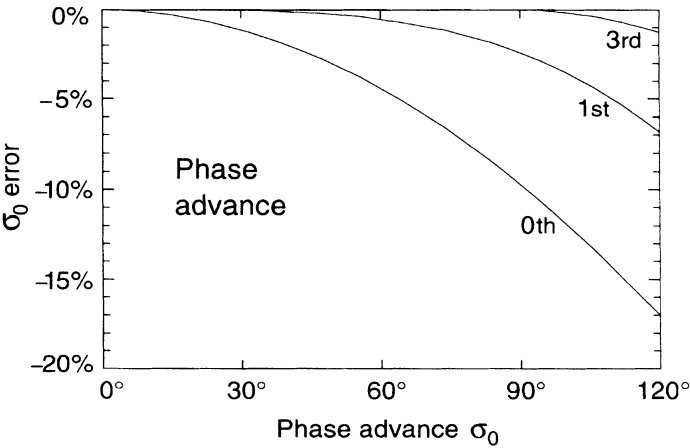


FIGURE 2: Accuracy of σ_0 to 0th order, 1st order, and 3rd order (cf. Figure 1). V_Q varies from 0 to 26.09 kV so that σ_0 ranges from 0° to 120.0° .